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A derivation of a new set of equations at the onset of the Bose–Einstein condensation

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Abstract

In this paper, we derive the equations characterizing the boundary layer which describes the transition of the distribution function of a gas of weakly interacting bosons to the distribution function of the gas in the presence of a Bose–Einstein condensate. To this end, we first rederive the classical Uehling–Uhlenbeck equation very briefly, taking as a starting point the dynamics of a system of many weakly interacting quantum particles. The solutions of the Uehling–Uhlenbeck equation yield blow up in finite time. Near the blow-up time, the approximations used to derive the Uehling–Uhlenbeck equation break down. We derive the set of equations that describe the building of correlations and the onset of quantum interference effects for the many-particle Hamiltonian system under the assumption that the blow-up for the Uehling–Uhlenbeck equation takes place in a self-similar form.

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1. Introduction

The kinetic equation that describes the evolution of a rarefied system of bosons was obtained by Nordheim [27] and by Uehling and Uhlenbeck [34]. In the limit of Born's approximation, the resulting equation, also known as the quantum Boltzmann equation, is the following:

$$\frac{\partial f}{\partial t} + \frac{p_1}{m}\nabla_x f(p_1, x, t) = C(f, f) \tag{1}$$

$$C(f,f) = \frac{4\pi g^2}{\hbar}\int \frac{\mathrm{d}p_2}{(2\pi\hbar)^3}\int \frac{\mathrm{d}p_3}{(2\pi\hbar)^3}\int \frac{\mathrm{d}p_4}{(2\pi\hbar)^3}(2\pi\hbar)^3\delta(\varepsilon(p_1)+\varepsilon(p_2)-\varepsilon(p_3)-\varepsilon(p_4)) \times q[f](p_1,p_2;p_3,p_4,t) \tag{2}$$

$$q[f](p_1,p_2;p_3,p_4,t) = [f(p_3)f(p_4)(1+f(p_1))(1+f(p_2)) - f(p_1)f(p_2)(1+(p_3))(1+f(p_4))] \tag{3}$$

with $g = 4\pi a\hbar^2/m$ where m is the mass of the particles, $\varepsilon(p) = |p|^2/2m$ is the energy of the particles and a is the s-wave scattering length (cf [40] for example).

The starting point for the derivation of a kinetic system of equations is a set of equations describing the dynamics of a system of N particles. In the case of weakly interacting bosons such dynamics is described by the Schrödinger equation for a system of N interacting particles. Assuming that the interaction between different particles is weak enough it is possible to obtain suitable evolution equations for the one-particle distribution function using a perturbative method. For classical particles, this has been mathematically proved for short times (cf [25]) or globally in time for special situations (cf [20]). This is the standard method used to derive the Uehling–Uhlenbeck equation (see for example the classical monographies [1, 3] as well as the more mathematically oriented approaches in [5, 7, 8, 31]). Similar arguments for the Fermionic case may be found in [19, 18].

It turns out that the solutions of (1)–(3) can develop singularities in finite time, as it has been obtained in the numerical simulations for spatially homogeneous distributions of particles in [29, 30] as well as in [24]. The interpretation of this blow-up phenomenon, given by the authors of these papers, is that such an event corresponds to the formation of the B–E condensate. As we will see in this paper, the derivation of the U–U equation, taking a quantum of many-particle systems as a starting point, is not valid near the time of the formation of the singularity.

On the other hand, the quantum dynamics of the particles in the condensate is described by the Gross–Pitaevskii equation (cf [2, 15, 16, 28, 40]). A rigorous mathematical proof of the precise formulation of this fact has been obtained in [26] for the stationary case and in [9] for the non-equilibrium case with short-range interactions in suitable scaling limits.

We are interested in the process formation of the condensate, a question which has already been considered by several authors, see for example [6, 12, 13, 22, 33]. In particular, in [6, 22] the dynamics of the particles of the system is approximated by means of a Gross–Pitaievskii equation with stochastic initial data. A similar problem is also studied in [12, 13]. Our main goal is to describe the transition between the kinetic regime described by the U–U equation and the quantum regime described by means of a nonlinear Schrödinger equation in a detailed manner. The relevant non-dimensional parameter is the quotient $\Delta E\Delta t/\hbar$, where ΔE is a characteristic value of the energy and Δt is a characteristic time scale for the variation of the density distributions. Interference effects cannot be ignored if this non-dimensional number is of order 1. We then derive the equation of the boundary layer which should describe in detail the transition from the distribution function at the critical time to the density function in which the condensate is present.

Some of the arguments presented in this paper are standard in statistical physics. For example, we make extensive use of the second quantification formalism in section 2. We also use the classical BBGKY hierarchy for quantum particles in section 3. The arguments used in these two sections are also used in the theory of weak turbulence and in general in the derivation of kinetic equations for weakly interacting waves (cf [4, 39]). Part of the arguments of section 4 are reminiscent of similar ones in [8, 40]. The main result of this paper is contained in sections 5 and 6. We have however included the results in the previous sections in order to fix the notation and explain as clearly as possible the limit under consideration.

2. The *N*-particles system and the second quantification formalism

We start recalling the classical derivation of the U–U equation that takes as a starting point the study of the dynamics of a quantum $N-$ particle system by means of the second quantification formalism. This will allow us to precise the assumptions in which such a derivation is based.

Let us assume that we have N quantum particles contained in a box $\Omega \equiv [0, L]^3$. We will denote the density of particles as $n = \frac{N}{L^3}$. We will also suppose that the particles interact by means of pair potentials. The Hamiltonian of the system is then given by

$$H_N = H_{0,N} + H_{1,N}, \tag{4}$$

where

$$H_{0,N} = -\frac{\hbar^2}{2m}\sum_{j=1}^{N}\Delta_{x_j}, \qquad H_{1,N} = \frac{1}{2}\sum_{k=1}^{N}\sum_{j=1,k\neq j}^{N} V(x_k - x_j). \tag{5}$$

The evolution of the system is given by means of Heisenberg's equation for the density matrix:

$$\mathrm{i}\hbar\partial_t\rho = [H_N, \rho] \tag{6}$$

$$\rho(0) = \rho_0. \tag{7}$$

The precise form of the initial density matrix that characterizes the initial state of the system will be given later. Let us precise the order of magnitude of the several parameters arising in the system. There are two main characteristic length scales in the problem, namely the average distance between particles $\mathrm{d} = \frac{L}{N^{1/3}}$ and the de Broglie length that is given by $\lambda = \frac{\hbar}{p_0} \equiv \frac{\hbar}{\sqrt{2mk_BT}}$. By assumption $p_0 \equiv \sqrt{2mk_BT}$ is just an estimate of a typical particle momentum. The temperature T is not a true thermodynamic temperature, because the system is not in equilibrium, but it is just a measure of the characteristic energy for the gas particles.

On the other hand, Born's approximation in mathematical terms means that we may assume that the interaction potential between particles V is smooth, but after deriving a set of kinetic equations we will take the limit $V \to g\delta(x)$, where g is defined just after formula (3).

The main assumptions on the physical parameters that we use in this paper are the following:

$$N \gg 1, \quad L \gg 1, \quad n = \frac{N}{L^3} = \text{constant} \tag{8}$$

$$\lambda \sim d \tag{9}$$

$$\frac{m\lambda^2 g}{d^3} \sim \frac{\lambda^2 a}{d^3} \ll 1. \tag{10}$$

Assumption (8) is the usual thermodynamic limit assumption that ensures that there are no boundary effects on the resulting equations. Assumption (10) means that the particle interactions are weak and allows us to derive a kinetic equation for the evolution of the distribution of particles. Finally, assumption (9) means that the particle densities are large enough to allow for the formation of the B–E condensate. This is related to the fact that the kinetic equation obtained under assumption (9) can yield blow-up in finite time.

We will first obtain a set of kinetic equations describing the evolution of the solutions of (6), (7) in the limit defined by (8)–(10). This problem was solved by Nordheim and Uehling and Uhlenbeck under the implicit additional hypothesis of the boundedness for the solution of the resulting kinetic equation. However, such assumption fails, because the solution of

the limit equation blows up in finite time as it has been seen in the numerical simulations of [24, 29, 30]. Therefore the Uehling–Uhlenbeck equation is not the correct limit for the system of particles under consideration in the limit (8)–(10), if the time t is close to the time of formation of a singularity. The main goal of this paper is to obtain a new kinetic equation describing the distribution of particles during the formation of the condensate.

2.1. Second quantification formalism

In order to study the N-particle system in the limit (8)–(10) we will use the formalism of the second quantification. Most of the computations in this subsection are standard, but we will reproduce them for the reader's convenience. We will assume that the Hamiltonian H_N in (4), (5) acts in the Hilbert space:

$$\mathcal{H}_N \equiv \bigotimes_{n=0}^{N} (L^2(\Omega))^n. \tag{11}$$

By definiteness we will assume that the wavefunctions satisfy periodic boundary conditions in Ω. Homogeneous Dirichlet boundary conditions would work similarly. For periodic boundary conditions, the eigenvalues of the momentum operator for a single particle $p_k \equiv -i\hbar\partial_{x_k}$ are given by

$$p = \frac{2\pi\hbar}{L}\ell, \qquad \ell \in \mathbb{Z}^3.$$

We will denote the normalized eigenfunctions associated with the operator $H_{0,N}$ as

$$|\ldots, n_\ell, \ldots\rangle, \qquad \ell \in \mathbb{Z}^3,$$

where n_ℓ is the number of particles in the state ℓ.

For notational convenience we will also use the following alternative way of writing these eigenfunctions:

$$|\ldots, n_\ell, \ldots\rangle = |n(\ell)\rangle, \qquad \text{where} \quad n : \mathbb{Z}^3 \to \mathbb{N} = \{0, 1, 2, \ldots\}$$

This notation will be convenient to write all the possible choices of occupation numbers in a short manner.

We introduce the well-known annihilation and creation operators a_ℓ, a_ℓ^+, N_ℓ whose action on these eigenfunctions is given by

$$a_\ell|\ldots, n_\ell, \ldots\rangle = \sqrt{n_\ell}|\ldots, n_\ell - 1, \ldots\rangle, \qquad \ell \in \mathbb{Z}^3 \tag{12}$$

$$a_\ell^+|\ldots, n_\ell, \ldots\rangle = \sqrt{n_\ell + 1}|\ldots, n_\ell + 1, \ldots\rangle, \qquad \ell \in \mathbb{Z}^3 \tag{13}$$

$$N_\ell|\ldots, n_\ell, \ldots\rangle \equiv a_\ell^+ a_\ell|\ldots, n_\ell, \ldots\rangle = n_\ell|\ldots, n_\ell, \ldots\rangle \tag{14}$$

and satisfy the commutation relations $\left[a_k, a_\ell^+\right] = \delta_{k,\ell}$, $[a_k, a_\ell] = \left[a_k^+, a_\ell^+\right] = 0$.

We now define the annihilation and creation operators of a particle at the point x of Ω by means of

$$\psi(x) = \frac{1}{L^{\frac{3}{2}}} \sum_{\ell\in\mathbb{Z}^3} a_\ell \, e^{\frac{2\pi i \ell x}{L}}, \qquad \psi^+(x) = \frac{1}{L^{\frac{3}{2}}} \sum_{\ell\in\mathbb{Z}^3} a_\ell^+ \, e^{-\frac{2\pi i \ell x}{L}}. \tag{15}$$

Note that using the commutation relations

$$[\psi(x), \psi^+(y)] = \frac{1}{L^3} \sum_{\ell\in\mathbb{Z}^3} e^{\frac{2\pi i \ell (x-y)}{L}} = \delta(x - y)$$

$$[\psi(x), \psi(y)] = [\psi^+(x), \psi^+(y)] = 0.$$

Using all these operators we can rewrite the operator $H_{0,N}$ as

$$H_{0,N} = \sum_{j=1}^{N} \frac{p_j^2}{2m} = \sum_{\ell \in \mathbb{Z}^3} \epsilon_\ell a_\ell^+ a_\ell,$$

where

$$\epsilon_\ell \equiv \frac{4\pi^2 \hbar^2 \ell^2}{2mL^2}, \qquad \ell \in \mathbb{Z}^3.$$

Taking the gradient of (15) we obtain

$$H_{0,N} = \frac{\hbar^2}{2m} \int_\Omega \nabla\psi^+(x) \nabla\psi(x)\, \mathrm{d}x. \tag{16}$$

On the other hand,

$$H_{1,N} = \frac{1}{2} \int_\Omega \mathrm{d}x_1 \int_\Omega \mathrm{d}x_2\, V(x_1 - x_2) \psi^+(x_1)\psi^+(x_2)\psi(x_1)\psi(x_2). \tag{17}$$

We define the distribution functions

$$f_{j,m}(x_1, \ldots, x_j; y_1, \ldots, y_m) \equiv \mathrm{Tr}(\rho \psi^+(y_1)\psi^+(y_2)\ldots\psi^+(y_m)\psi(x_1)\psi(x_2)\ldots\psi(x_j)) \tag{18}$$

The computation of the evolution equations for the functions $f_{j,m}$ from (6) is standard (see for example [1]) and gives

$$\mathrm{i}\hbar\partial_t f_{j,m} = \mathrm{Tr}(\rho[\psi^+(y_1)\psi^+(y_2)\ldots\psi^+(y_m)\psi(x_1)\psi(x_2)\ldots\psi(x_j), H_N]). \tag{19}$$

On the other hand, we can compute the commutator in (19) to obtain the following evolution equation for the distribution functions $f_{j,m}$

$$\begin{aligned}
\mathrm{i}\hbar\partial_t f_{j,m}(x_1, \ldots, x_j; y_1, \ldots, y_m) = &-\frac{\hbar^2}{2m}\left(\sum_{s=1}^{j} \Delta_{x_s} - \sum_{s=1}^{m} \Delta_{y_s}\right) f_{j,m}(x_1, \ldots, x_j; y_1, \ldots, y_m) \\
&+ \int_\Omega \mathrm{d}\xi \left[\sum_{k=1}^{j} V(\xi - x_k) - \sum_{k=1}^{m} V(\xi - y_k)\right] f_{j+1,m+1}(x_1, \ldots, x_j, \xi; y_1, \ldots, y_m, \xi) \\
&+ \frac{1}{2}\left[\sum_{k=1}^{j}\sum_{s=1,k\neq s}^{j} V(x_k - x_s) - \sum_{k=1}^{m}\sum_{s=1}^{m} V(y_k - y_s)\right] f_{j,m}(x_1, \ldots, x_j; y_1, \ldots, y_m).
\end{aligned} \tag{20}$$

2.2. *On the choice of the initial data*

In order to solve the system of equations (20) we must prescribe suitable initial data. We will assume that the initial matrix density $\rho(0)$ satisfies

$$\rho(0) = \rho_0 = \frac{1}{Q} \sum_{n:\mathbb{Z}^3 \to N} P_0(z, \Theta; n)|n\rangle\langle n| \tag{21}$$

$$P_0(z, \Theta; n) \equiv z^{[\sum_{\ell\in\mathbb{Z}^3} n(\ell)]} \left(\prod_{\ell\in\mathbb{Z}^3} (\theta_\ell)^{n(\ell)}\right) \tag{22}$$

$$n : \mathbb{Z}^3 \to \mathbb{N}, \qquad \sum_{\ell\in\mathbb{Z}^3} n(\ell) < \infty, \tag{23}$$

where

$$Q = \sum_{n:\mathbb{Z}^3 \to N} P_0(z, \Theta; n)$$

has been chosen in order to have $\mathrm{Tr}(\rho_0) = 1$ and where

$$\theta_\ell \equiv \Theta\left(\frac{2\pi^2\hbar^2}{mk_BT}\frac{\ell^2}{L^2}\right). \tag{24}$$

Choosing the initial data as in (21), (22) we assume that the particles are independently and homogeneously distributed in space according to the distribution $\Theta(\cdot)$ in the space of energy. Since we use a macrocanonical distribution the number of variables is a stochastic variable. In the thermodynamic limit, the fluctuations in the number of particles can be expected to disappear as it is usual in statistical physics. The value of z is chosen to obtain a given average number of particles N for the distribution. Therefore,

$$\langle N\rangle = \mathrm{Tr}(\rho_0 N) = \frac{1}{Q}\sum_{n:\mathbb{Z}^3\to N} P_0(z,\Theta;n)N(n) = z\frac{\partial(\log(Q))}{\partial z},$$

where $N(n) = \sum_{\ell\in\mathbb{Z}^3} n(\ell)$.

Instead of analyzing the original system (6), (7), we will study the equivalent system of equations (20) that is more convenient to use perturbative arguments. Due to (18) we must solve these equations with initial data:

$$f_{j,m;0}(x_1,\dots,x_j;y_1,\dots,y_m) \equiv \mathrm{Tr}(\rho_0\psi^+(x_1)\psi^+(x_2)\dots\psi^+(x_j)\psi(y_1)\psi(y_2)\dots\psi(y_m)). \tag{25}$$

Using (21), (22) we obtain

$$f_{j,m;0}(x_1,\dots,x_j;y_1,\dots,y_m) = 0 \qquad \text{if} \quad j\neq m. \tag{26}$$

The evolution equations (20), with initial data (26), admit a solution satisfying

$$f_{j,m}(x_1,\dots,x_j;y_1,\dots,y_m;t) = 0 \qquad \text{if} \quad j\neq m.$$

Therefore, we can restrict our study to the functions

$$F_k(x_1,\dots,x_k;y_1,\dots,y_k;t) \equiv f_{k,k}(x_1,\dots,x_k;y_1,\dots,y_k;t).$$

On the other hand, we can compute the initial distribution $F_{k,0}(x_1,\dots,x_k;y_1,\dots,y_k) \equiv F_k(x_1,\dots,x_k;y_1,\dots,y_k;0)$ that due to (15) and (25) is given by

$$F_{k,0}(x_1,\dots,x_k;y_1,\dots,y_k) = \frac{1}{L^{3k}}\sum_{\ell_1\in\mathbb{Z}^3}\dots\sum_{\ell_k\in\mathbb{Z}^3}\sum_{j_1\in\mathbb{Z}^3}\dots\sum_{j_k\in\mathbb{Z}^3} \mathrm{e}^{-\frac{2\pi\mathrm{i}(\ell_1x_1+\dots+\ell_kx_k)}{L}+\frac{2\pi\mathrm{i}(j_1y_1+\dots+j_ky_k)}{L}}$$
$$\times\,\mathrm{Tr}\big(\rho_0 a^+_{\ell_1}a^+_{\ell_2}\dots a^+_{\ell_k}a_{j_1}\dots a_{j_k}\big). \tag{27}$$

By (21) we then have

$$\mathrm{Tr}\big(\rho_0 a^+_{\ell_1}a^+_{\ell_2}\dots a^+_{\ell_k}a_{j_1}\dots a_{j_k}\big) = \frac{1}{Q}\sum_{n:\mathbb{Z}^3\to N}\prod_{\ell\in\mathbb{Z}^3}(z\theta_\ell)^{n(\ell)}\langle n|a^+_{\ell_1}a^+_{\ell_2}\dots a^+_{\ell_k}a_{j_1}\dots a_{j_k}|n\rangle.$$

Using standard statistical physics computations (cf [3]) we can approximate (27) in the limit $L\to\infty$ as

$$F_{k,0}(x_1,\dots,x_k;y_1,\dots,y_k) = \sum_{\sigma\in S^k}\prod_{m=1}^{k}F_0(y_m - x_{\sigma(m)};z), \tag{28}$$

where

$$F_0(y;z) \equiv \int_{\mathbb{R}^3}\left[\frac{z\Theta(\overline{\epsilon}(\xi))\,\mathrm{e}^{2\pi\mathrm{i}y\xi}}{1-z\Theta(\overline{\epsilon}(\xi))}\right]\mathrm{d}\xi, \qquad \overline{\epsilon}(\xi) \equiv \frac{2\pi^2\hbar^2}{mk_BT}\xi^2. \tag{29}$$

3. The small correlations approximation

3.1. Non-dimensional equations

Summarizing, we have reduced the problem to the following system of equations (cf (20), (28)):

$$\mathrm{i}\hbar\partial_t F_k(x_1,\dots,x_k;y_1,\dots,y_k,t) = A_1 + A_2 + A_3; \qquad k = 1,2,\dots \tag{30}$$

$$A_1 = -\frac{\hbar^2}{2m}\left(\sum_{s=1}^{k}\left[\Delta_{x_s} - \Delta_{y_s}\right]\right) F_k(x_1,\dots,x_k;y_1,\dots,y_k;t) \tag{31}$$

$$A_2 = \int_\Omega \mathrm{d}\xi \left[\sum_{j=1}^{k}[V(\xi - x_j) - V(\xi - y_j)]\right] F_{k+1}(x_1,\dots,x_k,\xi;y_1,\dots,y_k,\xi;t) \tag{32}$$

$$A_3 = \frac{1}{2}\left[\sum_{j=1}^{k}\sum_{s=1,j\neq s}^{k}[V(x_j - x_s) - V(y_j - y_s)]\right] F_k(x_1,\dots,x_k;y_1,\dots,y_k;t) \tag{33}$$

with initial data:

$$F_{k,0}(x_1,\dots,x_k;y_1,\dots,y_k) = \sum_{\sigma\in S^k}\prod_{m=1}^{k} F_0(y_m - x_{\sigma(m)};z). \tag{34}$$

The first two equations of this hierarchy are

$$\begin{aligned}\mathrm{i}\hbar\partial_t F_1(x_1;y_1;t) = {}& -\frac{\hbar^2}{2m}\left(\Delta_{x_1} - \Delta_{y_1}\right)F_1(x_1;y_1;t) + \int_\Omega \mathrm{d}\xi[V(\xi - x_1) - V(\xi - y_1)] \\ & \times [F_1(x_1;y_1;t)F_1(\xi;\xi;t) + F_1(x_1;\xi;t)F_1(\xi;y_1;t)] \\ & + \int_\Omega \mathrm{d}\xi[V(\xi - x_1) - V(\xi - y_1)]F_2(x_1,\xi;y_1,\xi;t)\end{aligned} \tag{35}$$

$$\begin{aligned}\mathrm{i}\hbar\partial_t F_2(x_1,x_2;y_1,y_2;t) = {}& -\frac{\hbar^2}{2m}\left(\Delta_{x_1} + \Delta_{x_2} - \Delta_{y_1} - \Delta_{y_2}\right)F_2(x_1,x_2;y_1,y_2;t) \\ & + \sum_{j=1}^{2}\int_\Omega \mathrm{d}\xi[V(\xi - x_j) - V(\xi - y_j)]F_3(x_1,x_2,\xi;y_1,y_2,\xi;t) \\ & + [V(x_1 - x_2) - V(y_1 - y_2)]F_2(x_1,x_2;y_1,y_2;t).\end{aligned} \tag{36}$$

In order to more clearly understand the limit that we are considering, we introduce the non-dimensional variables

$$x = \lambda\hat{x}, \quad V(x) = \frac{g}{\lambda^3}\widehat{V}(\hat{x}), \quad F_k(x) = \left(\frac{1}{d^3}\right)^k \widehat{F}_k(\hat{x}), \quad t = \frac{2m\lambda^2}{\hbar\epsilon^2}\hat{t}, \quad p = \frac{\hbar}{\lambda}\hat{p} \tag{37}$$

$$g = \epsilon\frac{\hbar^2}{2m\lambda^2}d^3, \tag{38}$$

where, due to (10), ϵ is a small parameter and by assumption the potential $\widehat{V}(\hat{x})$ is now of order 1. Our choice of time scale is due to the fact that we want to obtain, in the limit $\epsilon \to 0$, an equation in which the particle density varies in times $\hat{t}$ of order 1. Note that $(x, p) \to (\hat{x}, \hat{p})$

is not a canonical transformation but transforms a 'quantum cell' in the phase space of volume $\hbar$ into another cell of volume one.

For the sake of simplicity, we drop the hats in all the variables $\hat{x}$, $\hat{p}$, $\hat{t}$, $\widehat{F_k}$ and $\widehat{V}$ as it is customary in the computations of asymptotic expansions. The system (35)–(36) then becomes

$$\begin{aligned} \mathrm{i}\partial_t F_1(x_1; y_1; t) = &-\frac{1}{\epsilon^2}\left(\Delta_{x_1} - \Delta_{y_1}\right) F_1(x_1; y_1; t) + \frac{1}{\epsilon}\int_\Omega \mathrm{d}\xi [V(\xi - x_1) - V(\xi - y_1)] \\ &\times [F_1(x_1; y_1; t) F_1(\xi; \xi; t) + F_1(x_1; \xi; t) F_1(\xi; y_1; t)] \\ &+ \frac{1}{\epsilon}\int_\Omega \mathrm{d}\xi [V(\xi - x_1) - V(\xi - y_1)] F_2(x_1, \xi; y_1, \xi; t) \end{aligned} \tag{39}$$

$$\begin{aligned} \mathrm{i}\partial_t F_2(x_1, x_2; y_1, y_2; t) = &-\frac{1}{\epsilon^2}\left(\Delta_{x_1} + \Delta_{x_2} - \Delta_{y_1} - \Delta_{y_2}\right) F_2(x_1, x_2; y_1, y_2; t) \\ &+ \frac{1}{\epsilon}\sum_{j=1}^{2}\int_\Omega \mathrm{d}\xi [V(\xi - x_j) - V(\xi - y_j)] F_3(x_1, x_2, \xi; y_1, y_2, \xi; t) \\ &+ \frac{1}{\epsilon}\left(\frac{d}{\lambda}\right)^3 [V(x_1 - x_2) - V(y_1 - y_2)] F_2(x_1, x_2; y_1, y_2; t). \end{aligned} \tag{40}$$

3.2. Small correlations limit

Our aim is to obtain closure relations for the functions F_k by means of a perturbative argument.

Note that in the absence of potential the system of equations (39), (40) might be explicitly solved and the resulting solutions have the form:

$$F_k(x_1, \ldots, x_k; y_1, \ldots, y_k; t) = \sum_{\sigma \in S^k}\prod_{m=1}^{k} F_1(x_{\sigma(m)}, y_m; t). \tag{41}$$

Moreover, the function F_0 in (29) is invariant under spatial translations whence $F_0(x_1; y_1) = F_0(x_1 - y_1)$. Since the system of equations (30)–(33) are also invariant under spatial translations it follows that $F_1(x_1; y_1; t) = F_1(x_1 - y_1; t)$ for any $t > 0$.

Note that in this case we can consider the solutions of this form as 'uncorrelated' solutions, although in a strict mathematical sense the corresponding probability distributions are not uncorrelated. But the only correlations between particles whose distribution is given by (41) would be those due to the symmetry of the wavefunctions due to the bosonic character of the particles (cf the discussion in [3]). In any case the approximation (41) is a convenient starting point for the computation of the solutions of (30)–(34) in a perturbative manner. We define the correlation functions G_k by means of the identity

$$\begin{aligned} F_k(x_1, \ldots, x_k; y_1, \ldots, y_k; t) = &\, G_k(x_1, \ldots, x_k; y_1, \ldots, y_k; t) \\ &+ \widetilde{F}_k(x_1, \ldots, x_k; y_1, \ldots, y_k; t), \end{aligned} \tag{42}$$

where we have defined

$$\widetilde{F}_k(x_1, \ldots, x_k; y_1, \ldots, y_k; t) = \sum_{\sigma \in S^k}\prod_{m=1}^{k} F_1(x_{\sigma(m)}, y_m; t). \tag{43}$$

It is possible to derive a kinetic approximation for (35), (36) under the following small correlation assumptions:

$$|G_k| \ll \prod_{m=1}^{k} |F_1|. \tag{44}$$

Indeed, under this assumption we obtain, plugging (42) into (35)–(36),

$$\begin{aligned}
\mathrm{i}\partial_t F_1(x_1; y_1; t) = &-\frac{1}{\epsilon^2}\big(\Delta_{x_1} - \Delta_{y_1}\big)F_1(x_1; y_1; t) + \frac{1}{\epsilon}\int_\Omega \mathrm{d}\xi [V(\xi - x_1) - V(\xi - y_1)] \\
&\times [F_1(x_1; y_1; t)F_1(\xi; \xi; t) + F_1(x_1; \xi; t)F_1(\xi; y_1; t)] \\
&+ \frac{1}{\epsilon}\int_\Omega \mathrm{d}\xi [V(\xi - x_1) - V(\xi - y_1)]F_2(x_1, \xi; y_1, \xi; t). \qquad (45)
\end{aligned}$$

$$\begin{aligned}
\mathrm{i}\partial_t F_2(x_1, x_2; y_1, y_2; t) = &-\frac{1}{\epsilon^2}\big(\Delta_{x_1} + \Delta_{x_2} - \Delta_{y_1} - \Delta_{y_2}\big)F_2(x_1, x_2; y_1, y_2; t) \\
&+ \frac{1}{\epsilon}\sum_{j=1}^{2}\int_\Omega \mathrm{d}\xi [V(\xi - x_j) - V(\xi - y_j)]\widetilde{F}_3(x_1, x_2, \xi; y_1, y_2, \xi; t) \\
&+ \frac{1}{\epsilon}\left(\frac{d}{\lambda}\right)^3 [V(x_1 - x_2) - V(y_1 - y_2)]\widetilde{F}_2(x_1, x_2; y_1, y_2; t). \qquad (46)
\end{aligned}$$

The relative strength of the terms yielding correlations is of order ϵ. This explains why in equation (46) we have approximated F_2 and F_3 by $\widetilde{F}_2$ and $\widetilde{F}_3$, respectively. Note that we have kept all the terms in equation (45) and only terms of order $1/\epsilon$ or larger in (46).

We then compute the evolution equations for $F_1(x_1; y_1; t)$ and $G_2(x_1, x_2; y_1, y_2; t)$ using (42), the approximation (45)–(46), neglecting terms of order $\mathcal{O}(\epsilon^2)$ and using Born's approximation which in the variables that we are using reduces to

$$V(x) = \delta(x). \qquad (47)$$

We finally obtain

$$\begin{aligned}
\mathrm{i}\partial_t F_1(x_1; y_1; t) = &-\frac{1}{\epsilon^2}\big(\Delta_{x_1} - \Delta_{y_1}\big)F_1(x_1; y_1; t) \\
&+ \frac{1}{\epsilon}g[G_2(x_1, x_1; y_1, x_1; t) - G_2(x_1, y_1; y_1, y_1; t)] \qquad (48)
\end{aligned}$$

$$\begin{aligned}
\mathrm{i}\partial_t G_2(x_1, x_2; y_1, y_2; t) = &-\frac{1}{\epsilon^2}\big(\Delta_{x_1} + \Delta_{x_2} - \Delta_{y_1} - \Delta_{y_2}\big)G_2(x_1, x_2; y_1, y_2; t) \\
&+ \frac{1}{\epsilon}g[F_1(x_2; x_1; t)\widetilde{F}_2(x_1, x_1; y_1, y_2; t) - F_1(y_1; y_2; t)\widetilde{F}_2(x_1, x_2; y_1, y_1; t)] \\
&+ \frac{1}{\epsilon}g[F_1(x_1; x_2; t)\widetilde{F}_2(x_2, x_2; y_1, y_2; t) - F_1(y_2; y_1; t)\widetilde{F}_2(x_1, x_2; y_2, y_2; t)] \\
&+ \frac{1}{\epsilon}\left(\frac{d}{\lambda}\right)^3 g[\delta(x_1 - x_2) - \delta(y_1 - y_2)]\widetilde{F}_2(x_1, x_2; y_2, y_2; t). \qquad (49)
\end{aligned}$$

The invariance of the initial distribution $F_0(x_1; y_1)$ under spatial translations implies that, with a slight abuse of language, the solutions of (48), (49) have the form:

$$F_1(x_1; y_1; t) = F_1(x_1 - y_1; t) \qquad (50)$$

$$G_2(x_1, x_2; y_1, y_2; t) = G_2(x_1 - y_1, x_2 - y_1; 0, y_2 - y_1; t). \qquad (51)$$

Under these assumptions equations (48), (49) reduce to

$$\mathrm{i}\partial_t F_1(x_1 - y_1; t) = \frac{1}{\varepsilon}g[G_2(x_1, x_1; y_1, x_1; t) - G_2(x_1, y_1; y_1, y_1; t)] \qquad (52)$$

$$\mathrm{i}\partial_t G_2(x_1, x_2; y_1, y_2; t) = -\frac{1}{\varepsilon^2}\big(\Delta_{x_1} + \Delta_{x_2} - \Delta_{y_1} - \Delta_{y_2}\big)G_2(x_1, x_2; y_1, y_2; t) + \frac{1}{\varepsilon}Q[F_1], \tag{53}$$

where

$$\begin{aligned} Q[F_1](x_1, x_2; y_1, y_2; t) &= [F_1(x_2; x_1; t)\widetilde{F}_2(x_1, x_1; y_1, y_2; t) - F_1(y_1; y_2; t)\widetilde{F}_2(x_1, x_2; y_1, y_1; t)] \\ &\quad + [F_1(x_1; x_2; t)\widetilde{F}_2(x_2, x_2; y_1, y_2; t) - F_1(y_2; y_1; t)\widetilde{F}_2(x_1, x_2; y_2, y_2; t)] \\ &\quad + \left(\frac{d}{\lambda}\right)^3 [\delta(x_1 - x_2) - \delta(y_1 - y_2)]\widetilde{F}_2(x_1, x_2; y_2, y_2; t) \end{aligned} \tag{54}$$

Due to (34) we have

$$G_2(x_1, x_2; y_1, y_2; 0) \equiv 0. \tag{55}$$

The system of equations (52)–(55) will be our starting point for the description of the Bose gas in which we are interested. Note that it is a closed system of partial differential equations.

4. The problem in the phase space

The function that describes the one-particle density in the phase space in quantum problems is the Wigner transform of $F_1(x_1; y_1; t)$. Such a function is defined as

$$f_1(x, p; t) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} F_1(x + \zeta; x - \zeta; t)\,\mathrm{e}^{\mathrm{i}\zeta p}\,\mathrm{d}\zeta, \tag{56}$$

where the normalization constant in (56) has been chosen in order to have

$$\int f_1(x, p)\,\mathrm{d}x\,\mathrm{d}p = N.$$

In the spatially homogeneous case we have $F_1(x + \zeta; x - \zeta; t) = F_1(\zeta; t)$ due to (51). Therefore, (56) reduces to the Fourier transform

$$f_1(x, p; t) = f_1(p, t) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} F_1(\zeta; t)\,\mathrm{e}^{\mathrm{i}\zeta p}\,\mathrm{d}\zeta. \tag{57}$$

In order to obtain the evolution equation for $f_1(p, t)$ we then take the Fourier transform of (53), (54):

$$\mathrm{i}\partial_t f_1(p, t) = \frac{1}{(2\pi)^3\epsilon}\int_{\mathbb{R}^3} [G_2(\zeta, \zeta; 0, \zeta; t) - G_2(\zeta, 0; 0, 0; t)]\,\mathrm{e}^{\mathrm{i}\zeta p}\,\mathrm{d}\zeta. \tag{58}$$

On the other hand, we have the following Fourier representation for the function $G_2(x_1, x_2; y_1, y_2; t)$:

$$g_2(\xi_1, \xi_2; \eta_1, \eta_2; t) = \frac{1}{(2\pi)^{12}}\int_{(\mathbb{R}^3)^4} \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}y_1\,\mathrm{d}y_2\,\mathrm{e}^{\mathrm{i}(\xi_1 x_1 + \xi_2 x_2 - \eta_1 y_1 - \eta_2 y_2)}G_2(x_1, x_2; y_1, y_2; t), \tag{59}$$

$$G_2(x_1, x_2; y_1, y_2; t) = \int_{(\mathbb{R}^3)^4} \mathrm{d}\xi_1\,\mathrm{d}\xi_2\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2\,\mathrm{e}^{-\mathrm{i}(\xi_1 x_1 + \xi_2 x_2 - \eta_1 y_1 - \eta_2 y_2)}g_2(\xi_1, \xi_2; \eta_1, \eta_2; t). \tag{60}$$

Let us write

$$\begin{aligned} w(\xi_1, \xi_2; \eta_1, \eta_2; t) &= \frac{1}{(2\pi)^{12}}\int_{(\mathbb{R}^3)^4} \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}y_1\,\mathrm{d}y_2\,\mathrm{e}^{\mathrm{i}(\xi_1 x_1 + \xi_2 x_2 - \eta_1 y_1 - \eta_2 y_2)} \\ &\quad \times Q[F_1](x_1, x_2; y_1, y_2; t) \end{aligned} \tag{61}$$

Taking the Fourier transform of (53) and using (60) in (58) we obtain the following system of equations for f_1 and g_2 :

$$\mathrm{i}\partial_t f_1(p,t) = \frac{1}{\epsilon}\int_{(\mathbb{R}^3)^4} \mathrm{d}\xi_1\,\mathrm{d}\xi_2\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2[\delta(p-\eta_1)-\delta(p-\xi_1)]g_2(\xi_1,\xi_2;\eta_1,\eta_2;t) \tag{62}$$

$$\begin{aligned}\mathrm{i}\partial_t g_2(\xi_1,\xi_2;\eta_1,\eta_2;t) &= \frac{1}{\epsilon^2}[\varepsilon(\xi_1)+\varepsilon(\xi_2)-\varepsilon(\eta_1)-\varepsilon(\eta_2)]g_2(\xi_1,\xi_2;\eta_1,\eta_2;t)\\ &\quad+\frac{1}{\epsilon}w(\xi_1,\xi_2;\eta_1,\eta_2;t),\end{aligned} \tag{63}$$

where $w(\xi_1,\xi_2;\eta_1,\eta_2;t)$ is as in (54), (61) and the energy $\varepsilon(p)$ in the non-dimensional variables is $\varepsilon(p) = p^2$. To obtain a closed system for f_1, g_2 it only remains to compute $w(\xi_1,\xi_2;\eta_1,\eta_2;t)$ in terms of f_1. To this end note that (57) yields

$$F_1(\zeta;t) = \int f_1(p,t)\,\mathrm{e}^{-\mathrm{i}p\zeta}\,\mathrm{d}p. \tag{64}$$

Using (50) in the formula of $Q[F_1]$ and plugging the final expression into (61) we obtain, after some computations,

$$\begin{aligned}w(\xi_1,\xi_2;\eta_1,\eta_2;t) &= 2\delta(\xi_1+\xi_2-\eta_1-\eta_2)q[f](\xi_1,\xi_2;\eta_1,\eta_2,t)\\ q[f_1](\xi_1,\xi_2;\eta_1,\eta_2,t) &= \left[f_1(\eta_1)f_1(\eta_2)\left(\left(\frac{d}{2\pi\lambda}\right)^3+f_1(\xi_1)\right)\left(\left(\frac{d}{2\pi\lambda}\right)^3+f_1(\xi_2)\right)\right.\\ &\quad\left.-f_1(\xi_1)f(\xi_2)\left(\left(\frac{d}{2\pi\lambda}\right)^3+f_1(\eta_1)\right)\left(\left(\frac{d}{2\pi\lambda}\right)^3+f_1(\eta_2)\right)\right],\end{aligned} \tag{65}$$

where we have dropped the time dependence of the function f_1 on the right-hand side of (65). The solution g_2 to (63) is then

$$\begin{aligned}g_2(\xi_1,\xi_2;\eta_1,\eta_2;t) &= -\frac{2\mathrm{i}}{\epsilon}\delta(\xi_1+\xi_2-\eta_1-\eta_2)\int_0^t \mathrm{e}^{-\frac{\mathrm{i}}{\epsilon^2}[\varepsilon(\xi_1)+\varepsilon(\xi_2)-\varepsilon(\eta_1)-\varepsilon(\eta_2)](t-s)}\\ &\quad\times q[f_1](\xi_1,\xi_2;\eta_1,\eta_2;s)\,\mathrm{d}s,\end{aligned} \tag{66}$$

where we have used that $g_2(\cdot,\cdot;\cdot,\cdot;0)\equiv 0$ due to (55) and (59). The Dirac measure in (66) may be simplified if we define

$$g_2(\xi_1,\xi_2;\eta_1,\eta_2;t) = \delta(\xi_1+\xi_2-\eta_1-\eta_2)\varphi(\xi_1,\xi_2;\eta_1,\eta_2;t),$$

from where (66) gives

$$\varphi(\xi_1,\xi_2;\eta_1,\eta_2;t) = -\frac{2\mathrm{i}}{\epsilon}\int_0^t \mathrm{e}^{-\frac{\mathrm{i}}{\epsilon^2}[\varepsilon(\xi_1)+\varepsilon(\xi_2)-\varepsilon(\eta_1)-\varepsilon(\eta_2)](t-s)}\,q[f_1](\xi_1,\xi_2;\eta_1,\eta_2;s)\,\mathrm{d}s \tag{67}$$

Plugging (67) into (62) and using the symmetry of $q[f_1]$ with respect to its arguments ξ_1, ξ_2, η_1 and η_2 we finally obtain the following equation for f_1:

$$\begin{aligned}\partial_t f_1(p_1,t) &= \frac{4}{\epsilon^2}\int_0^t \mathrm{d}s\int_{(\mathbb{R}^3)^3}\mathrm{d}p_2\,\mathrm{d}p_3\,\mathrm{d}p_4\left\{\cos\left[\frac{1}{\epsilon^2}(\varepsilon(p_1)+\varepsilon(p_2)-\varepsilon(p_3)-\varepsilon(p_4))(t-s)\right]\right\}\\ &\quad\times\delta(p_1+p_2-p_3-p_4)q[f_1](p_1,p_2;p_3,p_4;s).\end{aligned} \tag{68}$$

Non-Markovian Boltzmann equations have been found in several physical situations (cf, for example, [2, 8, 14, 23, 40] and references therein).

5. The kinetic limit: the Uehling–Uhlenbeck equation

5.1. Derivation of the Uehling–Uhlenbeck equation

The formal derivation of the U–U equation would then proceed as follows. If we suppose that

$$\frac{1}{\epsilon^2}(\varepsilon(p)+\varepsilon(p_2)-\varepsilon(p_3)-\varepsilon(p_4))(\tau-\sigma)\gg 1 \tag{69}$$

a simple formal argument gives

$$\frac{1}{\epsilon^2}\cos\left[\frac{1}{\epsilon^2}(\varepsilon(p)+\varepsilon(p_2)-\varepsilon(p_3)-\varepsilon(p_4))(\tau-\sigma)\right] \rightharpoonup \pi\delta(\tau-\sigma)\delta(\varepsilon(p)+\varepsilon(p_2)-\varepsilon(p_3)-\varepsilon(p_4)) \tag{70}$$

in the sense of measures, where in that formula $p_2 \equiv p_3+p_4-p_1$. We finally end up with the U–U equation,

$$\partial_t f_1(p_1,t) = 4\pi\int_{(\mathbb{R}^3)^3} \mathrm{d}p_2\,\mathrm{d}p_3\,\mathrm{d}p_4\,\delta(\varepsilon(p_1)+\varepsilon(p_2)-\varepsilon(p_3)-\varepsilon(p_4)) \times\delta(p_1+p_2-p_3-p_4)q[f_1](p_1,p_2;p_3,p_4;t). \tag{71}$$

Note however that this approximation requires condition (69), which, using the original physical variables, can be formulated as the quasiclassical condition $\Delta E\Delta t \gg \hbar$. Equation (71) is just equations (1)–(3) written in a different system of units.

5.2. The loss of validity of the kinetic approximation

According to the blow-up scenario of Semikoz & Tkachev (cf [29, 30]) and Pomeau *et al* (cf [21, 24]) the blow-up for equation (71) takes place in a self-similar manner and the distribution of particles has relevant variations in the regions of the space of momentum p whose size rescales like the power $(T-t)^\beta$ for some positive β. In order to describe this region by means of an equation free of parameters we look for self-similar solutions of (71). Such solutions have the form

$$f(t,p) = (T-t)^{-2\beta-1/2}\Phi(\xi), \qquad \xi = \frac{p}{(T-t)^\beta}, \tag{72}$$

where the numerically computed value of β is such that $\beta = 1.069$ (see [24]). The function Φ, that is of order 1, solves then an integro differential equation free of parameters. Note also that the functional form (72) immediately tells us the time scales for which the interference effects in (68) cannot be ignored or, more precisely, in dimensional variables, the time scale where $\Delta E\Delta t \sim \hbar$. This happens if $p^2(T-t)\sim\epsilon^2$ or equivalently if

$$(T-t)\sim\frac{\epsilon^2}{p^2}. \tag{73}$$

Since $p\sim(T-t)^\beta$ in the self-similar region, we obtain that the interference effects appear at times

$$(T-t)\sim\epsilon^{\frac{2}{2\beta+1}}. \tag{74}$$

For this time scale we have to introduce a boundary layer in order to take into account the interference effects in (68) that have been neglected in (71).

5.3. The correlations of order 1 in the boundary layer

It turns out that in the same time scale where (70) starts failing, the small correlation approximation condition (44) ceases being valid. Indeed, assuming the self-similar behaviour (72) we obtain

$$F_1(\zeta, t) = (2\pi)^3 \int_{\mathbb{R}^3} f_1(p, t)\, \mathrm{e}^{-\mathrm{i}\zeta p}\, \mathrm{d}p, \tag{75}$$

$$= (2\pi)^3 (T-t)^{\beta-1/2} \int \Phi(Z)\, \mathrm{e}^{-\mathrm{i}\zeta (T-t)^{\beta} Z}\, \mathrm{d}Z, \tag{76}$$

$$= (T-t)^{\beta-1/2} \Psi(Z(T-t)^{\beta}). \tag{77}$$

On the other hand, (53) and (54) yield that for the boundary layer time scale

$$G \sim \frac{1}{\epsilon} F_1^3 (T-t)$$

from where, we obtain, using (74),

$$G \sim (T-t)^{2\beta-1} \sim F_1^2. \tag{78}$$

A similar argument shows that $|G_k| \sim \Pi_{m=1}^{k} |F_1|$ for $k > 1$. It then follows that the approximation of the system (35), (36) by system (45), (46) breaks down at the time scale (74).

6. The boundary layer: analytic description

In this section, we derive the set of equations describing the boundary layer where the kinetic approximation is lost. Since, as we have seen, the correlations are of order 1 in that region, we need to keep a major part of the equations in system (30). Using the non-dimensional variables (37) that system becomes

$$\mathrm{i}\partial_t F_k(x_1, \ldots, x_k; y_1, \ldots, y_k, t) = A_1 + A_2 + A_3; \qquad k = 1, 2, \ldots, \tag{79}$$

$$A_1 = -\frac{1}{\epsilon^2} \left(\sum_{s=1}^{k} \left[\Delta_{x_s} - \Delta_{y_s} \right] \right) F_k(x_1, \ldots, x_k; y_1, \ldots, y_k; t), \tag{80}$$

$$A_2 = \frac{1}{\epsilon} \int_{\Omega} \mathrm{d}\xi \left[\sum_{j=1}^{k} [V(\xi - x_j) - V(\xi - y_j)] \right] F_{k+1}(x_1, \ldots, x_k, \xi; y_1, \ldots, y_k, \xi; t) \tag{81}$$

$$A_3 = \frac{1}{2\epsilon} \left(\frac{d}{\lambda} \right)^3 \left[\sum_{j=1}^{k} \sum_{s=1, k \neq s}^{k} [V(x_j - x_s) - V(y_j - y_s)] \right] \times F_k(x_1, \ldots, x_k; y_1, \ldots, y_k; t). \tag{82}$$

The rescaling (74) suggests to define new variables as follows:

$$F_k(x_1, \ldots, x_k; y_1, \ldots, y_k; t) = \epsilon^{\frac{2\beta-1}{2\beta+1} k} H_k(X_1, \ldots, X_k; Y_1, \ldots, Y_k; \tau) \tag{83}$$

$$T - t = -\epsilon^{\frac{2}{2\beta+1}} \tau, \qquad x_i = \epsilon^{-\frac{2\beta}{2\beta+1}} X_i, \qquad p = \epsilon^{\frac{2\beta}{2\beta+1}} P. \tag{84}$$

Neglecting lower order terms in ϵ and using that $V(x) = \delta(x)$ we obtain that the functions H_k satisfy at leading order the following system:

$$\begin{cases} \mathrm{i}\partial_\tau H_k(X_1,\dots,X_k;Y_1,\dots,Y_k,\tau) = A_1 + A_2; \qquad k = 1,2,\dots \\ A_1 = -\left(\sum_{s=1}^{k}\left[\Delta_{X_s} - \Delta_{Y_s}\right]\right) H_k(X_1,\dots,X_k;Y_1,\dots,Y_k;\tau) \\ A_2 = \sum_{j=1}^{k}[H_{k+1}(X_1,\dots,X_k,X_j;Y_1,\dots,Y_k,X_j;\tau) \\ \quad - H_{k+1}(X_1,\dots,X_k,Y_j;Y_1,\dots,Y_k,Y_j;\tau)]. \end{cases} \tag{85}$$

This system must be solved with the asymptotic condition

$$H_1(X,Y,\tau) \sim (-\tau)^{\beta-1/2}\Psi((X-Y)(-\tau)^\beta) \qquad \text{as} \quad \tau \to -\infty \tag{86}$$

$$H_k(X_1,\dots,X_k;Y_1,\dots,Y_k;\tau) \sim \sum_{\sigma\in S^k}\prod_{m=1}^{k} H_1(X_{\sigma(m)},Y_m;\tau) \qquad \text{as} \quad \tau \to -\infty. \tag{87}$$

Note that formula (85) implies that all the correlation functions G_k defined in (42) were of the order of $\Pi_{m=1}^k|F_1|$ in the time scale (74).

This problem may also be expressed in the phase space using Wigner transforms that are defined as

$$\varphi_k(X_1,\dots,X_k;P_1,\dots,P_k;\tau) = \frac{1}{(2\pi)^3}\int \mathrm{d}\zeta_1\dots\mathrm{d}\zeta_k\,\mathrm{e}^{\mathrm{i}\sum_{j=1}^k\zeta_j P_j} \times H_k(X_1-\zeta_1,\dots,X_k-\zeta_k;X_1+\zeta_1,\dots,X_k+\zeta_k;\tau).$$

Plugging this into the system (85) we obtain

$$\begin{aligned} \frac{\partial\varphi_k}{\partial\tau} + \sum_{j=1}^{k} P_j\cdot\nabla_{X_j}\varphi_k = (2\pi)^{3k}\sum_{j=1}^{k}\int \mathrm{d}\zeta_j\,\mathrm{d}\tilde{P}_j\,d\tilde{P}_{k+1}\,\mathrm{e}^{\mathrm{i}\zeta_j(P_j-\tilde{P}_j)} \\ \times[\varphi_{k+1}(X_1,\dots,X_k,X_j-\zeta_j;P_1,\dot{s},\tilde{P}_j,\dots,P_k,\tilde{P}_{k+1};\tau) \\ -\varphi_{k+1}(X_1,\dots,X_k,X_j+\zeta_j;P_1,\dot{s},\tilde{P}_j,\dots,P_k,\tilde{P}_{k+1},\tau)]; \qquad k = 1,2,\dots. \end{aligned} \tag{88}$$

The asymptotic data as $\tau \to -\infty$ are now determined by

$$\varphi_1(X;P;\tau) = \varphi_1(P;\tau) \sim (-\tau)^{-\beta-1/2}\Phi\left(\frac{P}{(-\tau)^\beta}\right) \tag{89}$$

$$\begin{aligned} \varphi_k(X_1,\dots,X_k;P_1,\dot{s},P_k;\tau) \sim \frac{1}{(2\pi)^{3k}}\sum_{\sigma\in S^k}\int \mathrm{d}\zeta_1\cdots\mathrm{d}\zeta_k\,\mathrm{e}^{\mathrm{i}\sum_{j=1}^k\zeta_j P_j} \\ \times\prod_{m=1}^{k} H_1(X_{\sigma(m)}-\zeta_{\sigma(m)}-X_m-\zeta_m;\tau), \qquad k>1. \end{aligned} \tag{90}$$

Both systems, (85)–(87) and (88)–(90), are rather complicated objects to study that we do not consider in detail in this paper. However, the solution of this problem should provide a clear description on how the transition from the kinetic regime to the quantum-dominated and highly correlated regime takes place.

It is interesting to compute the time for which the correlations appear in physical variables. Using (74), we obtain that such a scale is given by

$$T^* - t = \frac{2m\lambda^2}{\hbar}\left(\frac{a\lambda^2}{d^3}\right)^{-\frac{4\beta}{2\beta+1}},$$

where T^* is the time at which the Uehling–Uhlenbeck equation blows up in the original physical units. For the kind of initial data that we are considering, given by (28) and (29), this time coincides with that obtained in [6]. This shows that we are actually looking at phenomena which occur in the same time scale.

The range of physical moments that would be described by the above systems ('in the boundary layer') is

$$p \sim \frac{\hbar}{\lambda}\left(\frac{a\lambda^2}{d^3}\right)^{\frac{2\beta}{2\beta+1}}.$$

Finally, the correlation lengths in this boundary layer are

$$x \sim \lambda\left(\frac{d^3}{a\lambda^2}\right)^{\frac{2\beta}{2\beta+1}}.$$

7. Concluding remarks

In this paper, we have derived the set of equations describing the quantum distribution of particles near the formation of the condensate. This has been achieved using the self-similar behaviour of the solutions of the Uehling–Uhlenbeck equation which develop singularities obtained in the numerical simulations of [21, 24, 29, 30]. In order to perform this derivation as precisely, we have briefly reviewed the derivation of the Uehling–Uhlenbeck model taking as a starting point a system of many quantum particles. This has also allowed us to obtain the time scale where such a derivation breaks down.

The system of equations that we have derived in this paper captures the type of mathematical object to be studied in order to describe the onset of interference effects near Bose–Einstein condensation, the way in which non-Markovian effects arise and the specific way in which correlations appear in the system. These questions have already been treated in the literature by several authors [6, 12, 13, 22, 33] in different ways. We think that it would be relevant to obtain a formalism describing Bose–Einstein condensation as mathematically rigorous as possible.

Our contribution is to deduce the set of equations (85)–(87), which do not contain any parameter, including the number of particles. A detailed knowledge of the properties of this system could shed new light on this physical process. It could be useful in particular to test some of the assumptions used in different approximations performed in the literature.

On the other hand, several problems where forming singularities, which would develop in finite time, are stopped from forming if additional effects are introduced, have been studied mathematically with great detail in recent years (cf [11, 37, 38]). Our purpose is to apply similar ideas to the problem of the Bose–Einstein condensation.

The mathematical problem (85)–(87) is still a formidable, although linear system of infinite equations. Analogous systems have been considered in cases where the distribution of particles is uncorrelated, for instance in the case of the derivation of Boltzmann equations (cf [31, 32]). The system that we have obtained in this paper is more complicated because it does not have uncorrelated solutions that might be written as the product of infinitely many one-particle distributions. Nevertheless, the study of systems with an infinite number of equations

has greatly advanced in recent years, cf [10, 17, 35, 36] for example. It makes sense therefore to state such a precise formulation of the mathematical problem to be considered in order to improve the understanding of the onset of the Bose–Einstein condensation.

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References

[1] Akhiezer A I and Peletminskij S V 1977 *Methods of Statistical Physics* (Moscow: Nauka) (in Russian)
Akhiezer A I and Peletminskij S V 1980 *Methods of Statistical Physics* (Elmsford, NY: Pergamon) (Engl. Transl.)
[2] Baier R and Stockkamp T 2004 Kinetic equations for Bose Einstein condensates from the 2PI effective action arXiv:hep-ph/0412310
[3] Balescu R 1975 *Equilibrium and Nonequilibrium Statistical Mechanics* (New York: Wiley)
[4] Balk A M and Zakharov V E 1998 Stability of weak-turbulence Kolmogorov spectra *Nonlinear Waves and Weak Turbulence (Applied Mathematical Sciences Translations Series 2* vol 182*)* ed V E Zakharov pp 1–81
[5] Benedetto D, Castella F, Esposito R and Pulvirenti M 2005 On the weak-coupling limit for bosons and fermions *Math. Mod. Meth. Appl. Sci.* **15** 1811–43
[6] Berloff N G and Svistunov B V 2002 Scenario of strongly nonequilibrated Bose–Einstein condensation *Phys. Rev.* A **66** 013603
[7] Cercignani C, Illner R and Pulvirenti M 1994 *The Mathematical Theory of Dilute Gases (Applied Mathematical Sciences* vol 106*)* (New York: Springer)
[8] Erdös L, Salmhofer M and Yau H T 2004 On the quantum Boltzmann equation *J. Stat. Phys.* **116** 367–80
[9] Erdös L, Schlein B and Yau H T 2005 Derivation of the cubic non-linear Schrödinger equation from quantum dynamics of many body systems arXiv:math-ph/0508010v2
[10] Escobedo M, Mischler S and Perthame B 2002 Gelation in coagulation and fragmentation models *Commun. Math. Phys.* **231** 157–88
[11] Escobedo M, Herrero M A and Velazquez J J L 1999 Radiation dynamics in a homogeneous plasma *Physica* D **126** 236–60
[12] Gardiner C W, Anglin J R and Fudge T I A 2002 The stochastic Gros–Pitaevskii equation *J. Phys. B: At. Mol. Opt. Phys.* **35** 1555–82
[13] Gardiner C W and Davis M J 2003 The stochastic Gross–Pitaevskii equation: II *J. Phys. B: At. Mol. Opt. Phys.* **36** 4731–53
[14] Gasenzer T, Berges J, Schmidt M G and Seco M 2005 Nonperturbative dynamical many-body theory of Bose–Einstein condensate *Phys. Rev.* A **72** 063604
[15] Ginzburg V L and Pitaevskii L P 1958 On the theory of superfluidity *Zh. Eksp. Teor. Fiz.* **34** 1240
Ginzburg V L and Pitaevskii L P 1958 On the theory of superfluidity *Sov. Phys.–JETP* **7** 858
[16] Gross E P 1961 Structure of a quantized vortex in boson systems *Nuovo Cimento* **20** 454
[17] Herrero M A, Velázquez J J L and Wrzosek D 2000 Sol–gel transition in a coagulation–diffusion model *Physica* D **141** 221–47
[18] Ho T G and Landau L J 1997 Fermi gas on a lattice in the van Hove limit *J. Stat. Phys.* **87** 821–45
[19] Hugenholtz N M 1983 Derivation of the Boltzmann equation for a Fermi gas *J. Stat. Phys.* **32** 231–54
[20] Illner R and Pulvirenti M 1986 Global validity of the Boltzmann quation for a two dimensional rare gas in vacuum *Commun. Math. Phys.* **105** 189–203
Illner R and Pulvirenti M 1989 Global validity of the Boltzmann quation for a two dimensional rare gas in vacuum *Commun. Math. Phys.* **121** 143–6 (erratum and improved result)
[21] Josserand C and Pomeau Y 2001 Nonlinear aspects of the theory of Bose–Einstein condensates *Nonlinearity* **14** R25–62
[22] Kagan Yu and Svistunov B V 1997 Evolution of correlations properties and appearance of broken symmetry in the process of Bose–Einstein condensation *Phys. Rev. Lett.* **79** 3331–4
[23] Kremp D and Bonitz M 1997 Non-Markovian Boltzmann equation *Ann. Phys.* **258** 320–59

[24] Lacaze R, Lallemand P, Pomeau Y and Rica S 2001 Dynamical formation of a Bose–Einstein condensate *Physica* D **152–153** 779–86
[25] Lanford III O 1975 *Time Evolution of Large Classical Systems (Lecture Notes in Physics* vol 38*)* ed E J moser (Berlin: Springer) pp 1–111
[26] Lieb E H and Seiringer R 2002 Proof of Bose–Einstein condensation for dilute trapped gases *Phys. Rev. Lett.* **88** 170409
[27] Nordheim L W 1928 On the kinetic method in the new statistics and its applications in the electron theory of conductivity *Proc. R. Soc.* A **119** 689–98
[28] Pitaevskii L P 1961 Vortex lines in an imperfect Bose gas *Zh. Eksp. Teor. Fiz.* **40** 646
Pitaevskii L P 1961 Vortex lines in an imperfect Bose gas *Sov. Phys.–JETP* **13** 451
[29] Semikov D V and Tkachev I I 1995 Kinetics of Bose condensation *Phys. Rev. Lett.* **74** 3093–97
[30] Semikov D V and Tkachev I I 1997 Condensation of Bosons in the kinetic regime *Phys. Rev.* D **55** 489–502
[31] Spohn H 1980 Kinetic equations from Hamiltonian dynamics: Markovian limits *Rev. Mod. Phys.* **53** 569–615
[32] Spohn H 1991 Large scale dynamics of interacting particles *Texts and Monographs in Physics* (Heidelberg: Springer)
[33] Stoof H T C 1999 Coherent versus incoherent dynamics during Bose–Einstein condensation in atomic gases *J. Low Temp. Phys.* **114** 11–108
[34] Uehling E A and Uhlenbeck G E 1933 Transport phenomena in Einstein-Bose and Fermi–Dirac gases *Phys. Rev.* **43** 552–61
[35] van Dongen P G J and Ernst M H 1985 Cluster size distribution in irreversible aggregation at large times *J. Phys. A: Math. Gen.* **18** 2779–93
[36] van Dongen P G J 1987 On the possible ocurrence of instantaneous gelation in Smoluchowski's coagulation equation *J. Phys. A: Math. Gen.* **20** 1889–904
[37] Velazquez J J L 2004 Point dynamics in a singular limit of the Keller–Segel model: 1. Motion of the concentration regions *SIAM J. Appl. Math.* **64** 1198–223
[38] Velazquez J J L 2004 Point dynamics in a singular limit of the Keller–Segel model: 2. Formation of the concentration regions *SIAM J. Appl. Math.* **64** 1224–48
[39] Zakharov V E 1984 Kolmogorov spectra in weak turbulence problems *Basic Plasma Physics* vol 2 ed A A Galeev and R Sudan (Amsterdam: North-Holland)
[40] Zaremba E, Nikuni T and Griffin A 1999 Dynamics of trapped Bose gases at finite temperatures *J. Low Temp. Phys.* **116** 277–345